Invited Lecture

Are You Really Teaching Mathematics? What Education Can Learn from History

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ABSTRACT It is believed that a knowledge of the history of mathematics could improve or expand an individual's understanding of the nature of mathematics, and hence may challenge teachers' epistemological beliefs of mathematics and, as a result, cause teachers to reconstruct their beliefs. Wilder reminds us that mathematics is a part of, and is influenced by, the culture in which it is found. As such, the culture dominates its elements, and in particular its mathematics. For instance, a Chinese mathematician living about the year 1200 C.E. would have mainly focused on computing with numbers and solving equations without paying attention to geometry as the ancient Greeks understood it. In contrast, a Greek mathematician of 200 B.C.E. would have focused more on geometrical proofs than on algebra and numerical computation as the Chinese practiced it. This paper aims to question the conventional view that treats mathematics as a significant instrument for developing one's personal career, instead advocating that we should regard mathematics as a cultural discipline of human endeavor in our teaching. I will interpret the history of mathematics in terms of a sociological macro-view and investigate the rise and fall of mathematics in the European and Chinese cultures to shed more light on the intellectual value of mathematics in education.

Keywords: History of mathematics; Mathematical culture; Teaching of mathematics.

1. Introduction

Mathematical knowledge is one of the oldest wisdoms of human beings. Both the Six Arts (rites, music, archery, charioteering, calligraphy, and mathematics) of ancient Chinese culture and the Quadrivium (arithmetic, geometry, music, and astronomy) of ancient Greek philosophy regard mathematics as a significant liberal art for educating a scholar. However, it was not unusual for distinct civilizations to have been devoted to scrutinizing an identical mathematical problem through their varied approaches. For instance, Archimedes asserts that the area of any circle is equal to a right-angled triangle in which one of the sides of the right angle is equal to the radius and the other to the circumference of the circle. Yet an ancient Chinese mathematical text, *Nine Chapters on the Mathematical Art* (九章算術), claims that multiplying half the circumference by half the diameter of the circle yields the area. With its useful

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applications in agriculture, natural sciences, engineering, and business, mathematics has become a fundamental subject in school (American Association for the Advancement of Science, AAAS, 1989). This glimpse into its history shows that mathematics can be seen as a locally-developed global language (Liu, 2017).

On the other hand, with the coming of the era of globalization, mathematics education has been obliged to respond to international trends and domestic needs. Therefore, it is treated as a globally-exchanged local practice. Few would deny that teachers are the most significant key persons contributing to the success or failure of the teaching mathematics in school. In what way and to what extent they interpret and transmit mathematical knowledge thus deserves attention. Not surprisingly, not all mathematics teachers hold an identical belief about what mathematics is and how it should be taught. As Thom (1973) proposed, "[A]ll mathematical pedagogy, even if scarcely coherent, rests on a philosophy of mathematics" (p. 204). Following this position, Hersh (1979) asserted, "[T]he issue then is not: what is the best way to teach? but, what is mathematics really all about?" (p. 34). In this paper, I will discuss the developmental nature of mathematics in terms of its micro history to investigate the effects of the history of mathematics on the teaching and learning of mathematics and, as well, to investigate the rise and fall of mathematics in the European and Chinese cultures from a sociological perspective. It is hoped that this approach may shed light on the intellectual value of mathematics and propose a potential philosophy for teachers' practices in teaching.

2. A Brief Review of the Development of Mathematics

The Greeks' logically deductive approach to mathematics dominates contemporary mathematical research. However, this was not the case thousands of years ago. Several other ancient civilizations, such as Babylonia, Egypt, Arabia, India and China, had been demonstrating highly developed mathematical knowledge in different ways. Prior to conducting a macro analysis of the development of mathematics, a brief review of ancient mathematics will be helpful.

2.1. Ancient Babylonian mathematics

Though there is a debate over the earliest appearance of the ancient Babylonian mathematics, its origin can be dated back to at least the third millennium B.C.E. Thanks to the hundreds of unearthed clay tablets in the Assyrian areas, we know that there had been a high-level investigation into the geometrical ratio in ancient Babylon. The clay tablets can be categorized into two kinds: problem texts and table texts. For instance, the clay tablet, YBC 7289 (Fig.1), contains a diagram and numbers. A sexagesimal 30 is inscribed along one edge of the square and sexagesimal sets of 1; 24, 51, 10 and 42; 25, 35 are written along the diagonal and in the lower segment of the square, respectively. The diagram and numbers have been decoded as follows:

$$1;\ 24,\ 51,\ 10 = 1 + \frac{24}{60} + \frac{51}{3600} + \frac{10}{216000} = 1.414212963 \doteq \sqrt{2}.$$

If we multiply “1; 24, 51, 10” by 30, we get

$$30\times 1.414212963 = 42.42638889 = 42+\frac{25}{60}+\frac{35}{3600} = 42;25,35.$$

Fig. 1. Babylonian clay tablet YBC 7289

The clay is believed to be used for measuring land during 1900-1600 B.C.E. Namely, if you own a square plot of land with 30 as its edge length, then its diagonal would be “42; 25, 35” long. It can be seen that the ancient Babylonians had a good understanding of the irrationals. Another clay tablet, Plimpton 322 (Fig. 2), has drawn much attention and debate about its use.

Fig. 2. Babylonian clay tablet Plimpton 322

Comprising four columns and 15 rows of numbers containing Pythagorean triples, the tablet is dated much earlier than any other civilization’s recorded insight into the triples. There is no consistent agreement about what the clay tablet was for (Britton, Proust and Shnider, 2011; Resnikoff and Wells, 1973). Recently, Mansfield and Wildberger (2017) claimed that Plimpton 322 is a table of Babylonian exact sexagesimal trigonometry, but were soon challenged by Lamb (2017). Regardless of the controversy it has caused, Plimpton 322 represents a significant achievement of ancient Babylonian mathematics.

2.2. Ancient Egyptian mathematics

Our current knowledge of Egyptian mathematics mostly relies on the *Rhind Papyrus* and *Moscow Papyrus*. The particular features of Egyptian mathematics are one, unit fractions wherein each fraction in the expression has a numerator equal to 1 and a denominator that is a positive integer, and two, its method of false assumption. For finding *aha,* the Egyptian word for the unknown quantity in an equation, hypothetical numbers were initially used to fit a simpler equation, followed by a revision of the hypothetical numbers to fit the original equation. For instance, the 24$^{\text{th}}$ problem in the Rhind Papyrus states, "*aha* and its one-seventh is 19. Find *aha.*" The method of false assumption starts by assuming that 7 is the hypothetical *aha*:

$$7+7\left(\frac{1}{7}\right)=8.$$

Multiply $2+\frac{1}{4}+\frac{1}{8}$ on both sides,

$$\left(2+\frac{1}{4}+\frac{1}{8}\right)\left(7+7\left(\frac{1}{7}\right)\right)=\left(2+\frac{1}{4}+\frac{1}{8}\right)8=19.$$

Then *aha* is

$$16+\frac{1}{2}+\frac{1}{8}.$$

The 14$^{\text{th}}$ problem of the Moscow Mathematical Papyrus, another ancient Egyptian papyrus containing mathematics, calculates the volume of a truncated pyramid: $V=\frac{1}{3}h(a^2+ab+b^2)$, where a and b are the base and top side lengths and h is the height. In particular, the ancient Egyptians knew how to apply the Pythagorean Theorem and method of false assumption to solve simultaneous equations. One of the two problems appeared in the *Berlin Papyrus*:

> "You are told the area of a square of 100 square cubits is equal to that of two smaller squares, the side of one square is 1/2 + 1/4 of the other. What are the sides of the two unknown squares?"

In modern terminology, it can be represented as follows:

$$\begin{cases} \dfrac{x}{y}=\dfrac{4}{3} \\ x^2+y^2=100, \end{cases}$$

where x and y are the lengths of the sides of the two squares.

We start by assuming that 3 and 4 are the two hypothetical side lengths. Since $3^2 + 4^2 = 5^2$ and $100 = 10^2 = (2 \cdot 5)^2$, we can multiply the length of both sides by 2^2, $6^2 + 8^2 = 10^2$. We can then assert that 6 and 8 are the actual lengths of sides x and y.

To determine the volume of a pyramid, ancient Egypt developed the method of false assumption, a unique technique in the history of mathematics, to solve quadratic simultaneous equations.

2.3. Ancient Greek mathematics

Though ancient Greek mathematics was grateful to the heritage of Babylonia and Egypt, the Greeks created a totally different culture and changed the nature of mathematics (Kline, 1962). Ancient Greek thinkers found the ways to apply mathematics to the fields of commerce and engineering, yet what impressed them most was the power of mathematical reasoning, the power of revealing the structure and nature of the physical world. According to Plato, mathematical objects are immaterial, just like God, goodness, courage and the human soul. Doing mathematics is the best way to understand the immaterial world. Plato asserted that the study of numbers facilitates the conversion of the soul itself from the world of generation to essence and truth, and an officer who had studied geometry would be a very different person from what he would be if he had not. Furthermore, based on the firm belief that the physical world is rationally designed, mathematics is the key to reveal the secret under the veil and, hence, astronomy became the chief scientific interest of the ancient Greeks.

Upon entering the Alexandrian period, a mixed interest in theoretical reasoning and practical investigation had risen. Euclid of Alexandria (born c. 325 B.C.E.), Aristarchus of Samos (310−230 B.C.E.), Eratosthenes of Cyrene (276−198 B.C.E.), Archimedes of Syracuse (287−212 B.C.E), and Claudius Ptolemy of Alexandria (100−170 C.E.) were the representative figures. Euclid's *Elements* synthesized previous known mathematical propositions and demonstrated them in a deductive fashion. On the basis of conventional astronomical phenomena and the basic calculation of geometrical objects, Aristarchus estimated the sizes of and the distance between the sun, Earth and moon, and Eratosthenes made a remarkably accurate estimation of the circumference of the earth. Archimedes expertly employed intuitive thinking and theoretical reasoning, and as well, skillfully combined physical principles and mathematical propositions to derive and rigorously prove a range of mathematical theorems. Some of his works can be found in the so-called Archimedes' palimpsest. Ptolemy's *Almagest*, a construction of a geocentric model of the universe, was the most influential mathematical and astronomical treaties until the appearance of Copernicus' *On the Revolutions of the Celestial Spheres* in 1543. Though ancient Greeks made little contribution to the study of numbers and the solving of equations, their mathematics not only reached the peak of the world at that time but also established the modern paradigm of mathematics.

2.4. Ancient Indian mathematics

Our knowledge about the mathematics of ancient India is mostly based on the scripts written in Sanskrit, a language used before the middle of the first millennium. However, unlike the aforementioned clay tablets in Babylon and papyrus in Egypt, very few original sources can be traced to reconstruct their mathematical knowledge with certainty. The first known texts written in Sanskrit are the "Vedas" (literally "knowledge"), which is a canon of hymns for religious ritual. The *Śulbasūtras* (literally "*rope-rules*") Vedic texts are the only sources of Indian mathematics during the Vedic period. Because altar construction usually requires doing area-preserving transformation, the geometric procedures were connected with sacrificial ritual in this manner. In one of the *Śulbasūtras*, called *Baudhāyana Śulbasūtras* (the *Śulbasūtras* texts are associated with the author's name), the process of transformation between circle and square were addressed as follows (cited in Katz, 2007):

- If it is desired to transform a square into a circle, [a cord of length] half the diagonal [of the square] is stretched from the centre to the east [a part of it lying outside the eastern side of the square]; with one-third [of the part lying outside] added to the remainder [of the half diagonal], the [required] circle is drawn.
- To transform a circle into a square, the diameter is divided into eight parts; one [such] part after being divided into twenty-nine parts is reduced by twenty-eight of them and further by the sixth [of the part left] less than the eighth [of the sixth part].

The above techniques pertaining to the circulature of a square imply the value of π to be 3.088. But this value is not consistent throughout. It can be found that π is approximated by other values elsewhere. The value was, therefore, obtained empirically, without a systematic approach.

In 327 B.C.E., Alexander the Great conquered some small kingdoms of northeastern India and started to spread Greek influence into this ancient civilization. In spite of the constant ups and downs among the different kingdoms in this land and despite the fact that Alexander's ambition ended with his premature death, the study of astronomy was always encouraged, triggering the development of trigonometry (Katz, 1998). The earliest known Indian text involving trigonometry is *Paitāmahasiddhānta*, written in about the early fifth century and containing a table of half-chords (*jyā-ardha*, Fig. 3). Note that the half-chord (sin *a*) in ancient India was different from the contemporary concept. We define sin *a* as the ratio of the line segment to the radius, but ancient Indians thought of sin *a* as the line segment itself. Following the study of trigonometry, the techniques of approximation and solving indeterminate equations were developed. India went on to achieve its mathematical peak in the 12th century and maintained it until the mid-16th century.

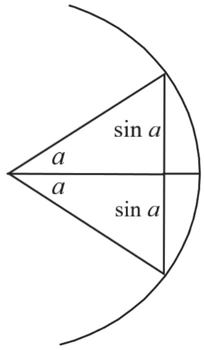

Fig. 3. The half-chords in *Paitāmahasiddhānta*

2.5. Ancient Chinese mathematics

Zhoubi Suanjing (周髀算經), *Arithmetical Classic of the Gnomon and the Circular Paths of Heaven*) is one of the oldest mathematical books dedicated to astronomical observations and calculations in ancient China (ca. 100 B.C.E.). It addresses a special case of the Shang Gao Theorem (商高定理), the Chinese version of the Pythagorean Theorem) and implicitly shows a general proof. The book begins with a conversation between the Duke of Zhou (周公) and the mathematician Shang Gao about the method for using *Bi* (髀 gnomon) to measure the width of the land and the height of the sky. Though the methodology was moderate and Shang Gao's mathematical reasoning was appropriate, the results were not all correct due to being based on a canopy heaven cosmological model, an umbrella-like heaven that rotates about a vertical axis rooted on a flat earth, which was adopted at the time. More than just the Shang Gao Theorem, the *Zhoubi Suanjing* is a collection of various ancient astronomical texts. However, the compilers of this book could have modified the original data (Li and Sun, 2009). Despite its flaws, this book deserves the title of 'the principal surviving document of early Chinese science' (Cullen, 1996), the earliest paradigm for demonstrating China's use of mathematical methods in astronomy.

In addition to astronomy, acoustics and optics were other branches of physics well studied in ancient China. *Guanzi* (管子), an ancient text traditionally attributed to the philosopher Guan Zhong (管仲, ca. 7th century B.C.E.), proposed the Method of Subtracting and Adding Thirds (三分損益法) to create musical scales, similar to the Pythagorean tuning system. *Mojing* (墨經, the *Mohist Canon* written by Mozi (墨子) and his followers, ca. 4th century B.C.E., Fig. 4) sequentially claimed eight propositions of optics for describing the phenomena of light, shadow, and pinhole imaging. It should be noted that, among all schools of philosophical thought in ancient China, Mohism (墨家) is unique in its inclusion of a discourse on mathematics and mechanics. According to Mozi, the reason for something is what must be before it will come about. There are two kinds of reasons: minor reasons and major reasons. The definition of a minor reason is "*having this, it will not necessarily be so; lacking this, necessarily it* *will not be so*” (小故，有之不必然，無之必不然). Obviously, the minor reason is the necessary condition in terms of modern logic. Major reason, on the other hand, is “*having this, it will necessarily be so; lacking this, necessarily it will not be so* (大故，有之必然，無之必不然), which are the sufficient and necessary conditions. The Mojing also demonstrates a Euclidean style of thought in defining dimension as “having something which it is bigger” (厚有所大也), circular as ‘having the same lengths from one center” (圜，一中同長也), and point as “the unit without dimension which precedes all others” (端，體之無序而最前者也). The principle of leverage was also discussed in the *Mojing* for interpreting the function of moving heavy objects in the steelyard, about 200 years earlier than Archimedes. However, the lack of quantitative analysis makes it impossible to carry out a mathematical discourse. Mohism was a very influential school of thought during the Warring States period (戰國時期) and was the largest rival to Confucianism (非儒即墨). However, Mohism was almost forgotten due to the Qin Dynasty’s promotion of Legalism (法家) and the Han Dynasty’s promotion of Confucianism.

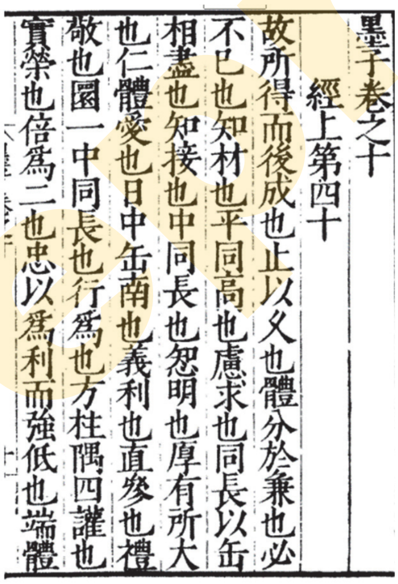
墨子卷之十
經上第四十
故所得而後成也止以久也體分於兼也必
不已也知材也平同高也慮求也同長以缶
相盡也知接也中同長也恕明也厚有所大
也仁體愛也日中缶南也義利也直參也禮
敬也圜一中同長也行爲也方柱隅四讙也
實榮也倍爲二也忠以爲利而強低也端體

Fig. 4. Mohist Canon

The most famous and influential ancient Chinese mathematical book is the *Jiuzhang Suanshu* (九章算術, *Nine Chapters on the Mathematical Art*). The author and original date of the book are unknown but it is estimated to have been written shortly after 200 B.C.E. Actually, in some sense, the *Jiuzhang Suanshu* is less like a mathematical treatise and more like a how-to reference manual. It presented 246 problems in life, business, and measurement, followed by answers and algorithms but without formal proof or derivation. In ancient China, Liu Hui (劉徽, 225−295) and Zu Chongzhi (祖沖之, 429−500) were the two most significant mathematicians prior to the Tang and Song dynasties. Liu Hui's major contribution is his commentary on the *Jiuzhang Suanshu*, demonstrating a typical Chinese style of inductive argumentation. Zu Chongzhi is well-known by his world-leading approximation of π. The Tang dynasty (618−907) government recruited and trained officers to do practical mathematics including measuring, taxation, and calendar making. Though the Tang government compiled and corrected the *Ten Mathematical Canons* (算經十書) as the official mathematical texts for imperial examinations, the mathematical texts studied by these imperial officers included problems and skills for solving problems without dealing with any new methods. "Thus there was no particular incentive for mathematical creativity" (Katz, 1998, p. 193).

3. A Cultural Survey

Fig. 5 is a chart graphing the ratio of GDP of all major powers from the year one A.D. to 2017 (Visual Capitalist, 2017). The wider the color band, the stronger the economy. It appears that, prior to the rise of the European Renaissance in the 14th century, China and India were the two greatest economic powers and, coincidentally, the achievements in mathematics of both civilizations reached their peaks during this period, respectively. Following the Tang dynasty, the Song dynasty (960−1279) is regarded as the golden or the greatest age of China (Fairbank, 1992; Stavrianos, 1971) for its high cultural achievement, and has even been named 'the Eastern Renaissance' (Miyazaki, 1950). The contribution made by several mathematicians during the Song

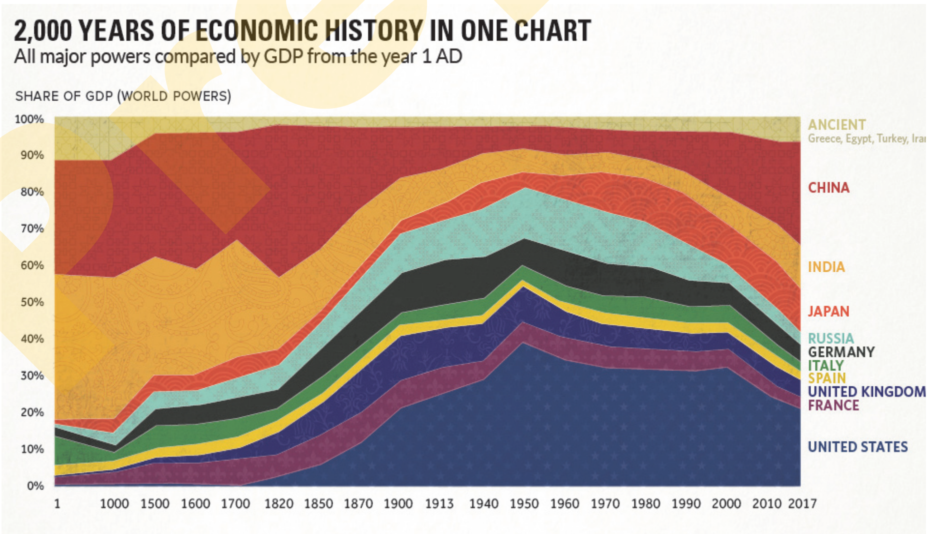

Fig. 5. 2,000 years of economy history (Visual Capitalist, 2017)
https://www.visualcapitalist.com/2000-years-economic-history-one-chart/

marks the peak of ancient Chinese mathematics, such as Jia Xian's (賈憲) method for extracting the square and cubic roots, Qin Jiushao's (秦九韶) Chinese Remainder Theorem, Li Ye's (李冶) techniques for solving polynomial equations, and Zhu Shijie's (朱士傑) method for solving high-order simultaneous equations of several variables. Despite its high achievement in computational arithmetic and instrumental techniques, the failure to use mathematics to reveal physical laws is one reason why ancient Chinese mathematics and science stagnated. This could be attributed to traditional Chinese philosophy which regarded the whole universe as an organic and self-sufficient system "in which there was no room for Laws of Nature, and hence, no fixed regularities to which it would be profitable to apply mathematics in the mundane sphere" (Needham, 1956, p. 325).

Besides, Brahmagupta proposed a general form of Heron's formula for the area of cyclic quadrilaterals. Following Āryabhaṭa and Brahmagupta's mathematical tradition, Bhāskara II studied the solution of quadratic, cubic and quartic indeterminate equations. He also proposed preliminary concepts of infinitesimal calculus and integral calculus as early as the 12th century.

Fig. 5 also shows that the economy of China was getting stronger again during 1700~1820, which was the period under the reign of the emperors Kangxi (康熙), Yongzheng (雍正), Qianlong (乾隆), and Jiaqing (嘉慶). During that time, China eagerly welcomed Western mathematics and sciences, and developed its own mathematical community. The aforementioned phenomena suggest a link between the development of mathematics and the degree of economic growth or cultural openness.

Raymond Wilder put forward the concept of mathematical culture specifically in the early days. He gave a speech on the cultural basis of mathematics at the International Congress of Mathematicians in 1950, expounding on the connotation and importance of mathematical culture. He claimed that he believed that only by recognition of the cultural basis of mathematics would a better understanding of its nature be achieved (Wilder, 1950, p.259). Because diverse mathematical practices developed and evolved in different civilizations in response to common problems that were encountered within a cultural context, Hersh (1997) expressed a cultural view that "mathematics must be understood as a human activity, a social phenomenon, part of human culture, historically evolved, and intelligible only in a social context" (p. xi).

To explore mathematics in culture is to understand the macroscopic development of mathematical knowledge. The first dawn of mathematical development came from the investigation of nature. Afterwards, due to the uniqueness of the evolutionary methods and processes of various human civilizations and societies, different rational paths and thinking cultures were created, and a "homogeneity and heterogeneity" of mathematical knowledge was born. Mathematics has been influenced by agriculture, commerce, industry, warfare, engineering, philosophy, psychology, and astronomy, and an understanding of mathematics requires one to take these key factors into account (Struik, 1948). The invention of typography in Europe contributed to the dissemination of mathematical knowledge during the Renaissance, which not only contributed to the mathematization of science in the 16th and 17th centuries, but even triggered the scientific revolution. However, this relationship between mathematical knowledge and social culture in the development of mathematics was not an inevitable trend. China, where typography originated earlier, did not have a similar developmental path. Rather, its development is related to the mathematical traditions of various cultures. Wilder (1950) reminds us of the interactive relationship between internal and external tensions between mathematics and other disciplines. In addition to the influence of the host culture, cultural infiltration among different ethnic groups may also lead the direction of mathematics in new directions. For instance, the ancient Greek mathematics was influenced by ancient Egypt and Babylonia, which then influenced the mathematics of Arabia and India. After that, because of the advantage of its symbolic system in operation and abstraction, the style of Western mathematics became mainstream, resulting in the gradual disappearance of cultural differences in contemporary mathematics.

4. Conclusion

As asked in the title of this paper, are you really teaching mathematics? What can education learn from history? In the book “Mathematics in Western Culture”, Kline (1954) demonstrated that mathematics is a subculture of the entire human culture. Actually, his larger attempt was to reveal that mathematics is “a major cultural driving force in Western civilization” (p. ix). This seemingly weird claim will of course attract criticism, but Kline believes this is due to a long-standing public misunderstanding of the nature of mathematics. He maintains that mathematics, although a body of knowledge, does not contain truths. Science is indeed pursuing the truth of the physical world, and mathematics just acts as a beacon, guiding science to its purpose. In addition to society’s need for a response to its problems, “over and above all other drives to create is the search for beauty” (p. 5).

The purpose of this study is to maintain that mathematics is not only an educational product but also a cultural creation. Mathematics problems arose from culture; the culture of mathematics then developed to become a part of mathematical knowledge, a knowledge which further influences other fields and forms other cultures. The interaction between mathematics and culture is not only related to mathematical knowledge itself but invisibly affects the reality of mathematics education. If the general public see mathematics only as a tool for solving problems, this view will mislead the public's understanding of the nature of mathematics. As Burton (2009) pointed out, the orientation of mathematics in a regional culture may constitute a barrier for certain ethnic groups to enter the domain of mathematics, and also shape the public's understanding of the culture of mathematics. Therefore, Burton believes that it is necessary to pay attention to the socio-political attitudes, values and behaviors towards mathematics in a society. We must see how to teach mathematics in a new light, namely, that the teaching of mathematics is a cultural inquiry as well as an educational issue.

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